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APPLICATION OF THE BURGERS' EQUATION WITH A VARIABLE COEFFICIENT TO THE STUDY OF NONPLANAR WAVE TRANSIENTS

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We consider the problem of constructing a model equation, i. e. the Burgers' equation, for the wave processes in a thermoelastic medium in the presence of cylindrical and spherical symmetry, and give a solution to the boundary value problem for the initial system of equations.

The Burgers' equation [1 – 4] serves as the model equation for a medium with dissipative properties. A solution of the Burgers' equation describing the motion in the Cartesian coordinate system was studied in detail in [5]. The cases of cylindrical and spherical symmetry however present definite difficulties.

1. Derivation of Burgers' equation with variable coefficients. We consider a process of deformation characterized by the relations

$$x^1 = X^1 + u^1 (X^1, t), \quad x^2 = X^2, \quad x^3 = X^3$$

where X^k and x^k $(k = 1, 2, 3)$ are the Lagrangian and Eulerian variables, respectively. The initial equations consist of the laws of conservation of impulse and energy for a continuum, written in a differential form in the Eulerian variables [6, 7]. We write these equations in the Lagrangian coordinates, taking into account the relations connecting the expressions for the physical quantities in the Lagrangian and Eulerian variables, respectively. We shall use the Lagrangian coordinates in all the arguments that follow. The five-constant theory of thermoelasticity will serve as the mathematical model of the present problem. Following [7], let us introduce the dimensionless quantities. The equations of the mathematical model of nonlinear thermoelasticity then become

$$\left[1+3(1+m_0)\,\varepsilon\frac{\partial U}{\partial \xi}+\varepsilon n m_1\frac{U}{\xi}-\varepsilon\theta\right]\frac{\partial^2 U}{\partial \xi^2}+ \tag{1.1}$$
$$n\left(1+\varepsilon m_2\frac{\partial U}{\partial \xi}+\varepsilon m_3\frac{U}{\xi}U-\varepsilon\theta\right)\frac{1}{\xi}\frac{\partial U}{\partial \xi}-$$
$$n\left(1+\varepsilon m_4\frac{U}{\xi}-\varepsilon\theta\right)\frac{1}{\xi}\frac{U}{\xi}-\left(1+\varepsilon\frac{\partial U}{\partial \xi}\right)\frac{\partial \theta}{\partial \xi}=\frac{\partial^2 U}{\partial \tau^2}$$
$$\frac{1}{\omega}\frac{\partial P}{\partial \xi}+\frac{n}{\omega}\left(1+\varepsilon\frac{\partial U}{\partial \xi}\right)\frac{P}{\xi}=$$
$$(1+\theta)\frac{\partial \theta}{\partial \tau}+e(1+\theta)\left(1+\varepsilon\frac{\partial U}{\partial \xi}\right)\frac{\partial^2 U}{\partial \xi\partial \tau}+ne(1+\theta)\left(1+\varepsilon\frac{U}{\xi}\right)\frac{1}{\xi}\frac{\partial U}{\partial \xi}$$
$$\beta\frac{\partial P}{\partial \tau}+P=\frac{\partial \theta}{\partial \xi}$$

where ω, β and e are the dimensionless diffusion, relaxation and coupling parameters, respectively, ε is a small parameter, m_i $(i=0,1,\ldots,4)$ are dimensionless moduli of elasticity and $n=0,1,2$ for the Cartesian, cylindrical and spherical coordinates, respectively.

The initial conditions for the system (1.1) are assumed to be null, and the boundary conditions are taken in the form

$$\left.\frac{\partial U(\xi,\tau)}{\partial \xi}\right|_{\xi=\xi_A}=q\varphi_1(\tau),\qquad \theta(\xi,\tau)\,|_{\xi=\xi_A}=0$$

where $\varphi_1(\tau)$ is a given continuous function.

In what follows, we shall use simultaneously the perturbation method and the coordinate transformation method. To do this, we expand the dependent variables into series in small parameter and introduce a semi-characteristic transformation of the independent variables in the form $\xi_1=(1+e)^{1/2}\tau-\xi,\ \tau_1=\varepsilon\xi$

In accordance with the perturbation method we obtain the following equation in the first approximation:

$$\frac{\partial V_0}{\partial \tau_1}+\frac{n}{2\tau_1}V_0+a_1V_0\frac{\partial V_0}{\partial \xi_1}=b_2\frac{\partial^2 V_0}{\partial \xi_1^2} \tag{1.2}$$

where

$$V_0=\partial U_0/\partial\xi_1,\quad a_1={}^3/_2\,(1+m_0+e)(1+e)^{-1}$$
$$a_2={}^1/_2\,e\,(\varepsilon\omega)^{-1}(1+e)^{-3/2}$$

We solve Eq. (1.2) with the initial condition

$$V_0(\xi_1,\tau_1)\,|_{\tau_1=t_A}=-q\varphi_1(\xi_1),\quad t_A=\varepsilon\xi_A,\quad q=\text{const}$$

Solution of (1.2) in this form presents definite difficulties, therefore following [1] we introduce additional transformations. Let us consider separately the cases with different values of n.

Plane wave, $n=0$. In this case (1.2) yields the classical Burgers' equation

$$\frac{\partial V_0}{\partial \tau_1}+a_1V_0\frac{\partial V_0}{\partial \xi_1}=a_2\frac{\partial^2 V_0}{\partial \xi_1^2}$$

the solution of which is known [5].

Cylindrical wave, $n = 1$. We introduce new variables

$$v_0 = V_0 (\tau_1)^{1/2}, \qquad \tau_2 = 2\tau_1^{1/2}$$

Then (1.2) yields

$$\frac{\partial v_0}{\partial \tau_2} + a_1 v_0 \frac{\partial v_0}{\partial \xi_1} = a_2^\circ (\tau_2) \frac{\partial^2 v_0}{\partial \xi_1^2} \tag{1.3}$$

where

$$a_2^\circ (\tau_2) = {}^1/_2 a_2 \tau_2 \tag{1.4}$$

Equation (1.3) is solved with the initial condition

$$v_0 (\xi_1, \tau_2) |_{\tau_2 = t_B} = - t_A^{1/2} q \varphi_1 (\xi_1), \qquad t_B = 2 t_A^{1/2}$$

Spherical wave, $n = 2$. We introduce new variables

$$v_0 = V_0 \tau_1, \qquad \tau_2 = \ln \tau_1$$

This yields (1.3) in which

$$a_2^\circ (\tau_2) = {}^1/_2 \, a_2 \exp \tau_2 \tag{1.5}$$

and the initial condition has the form

$$v_0 (\xi_1, \tau_2) |_{\tau_2 = t_B} = - t_A q \varphi_1 (\xi_1), \qquad t_B = \ln t_A$$

2. Solution of the Burgers' equation with variable coefficients. Investigation of one-dimensional waves in a nonlinear thermoelastic medium can be reduced to the Burgers' equation (2.1) and the initial conditions (2.2)

$$\frac{\partial v_0}{\partial t} + a_1 v_0 \frac{\partial v_0}{\partial y} = a_2 (t) \frac{\partial^2 v_0}{\partial y^2} \tag{2.1}$$

$$v_0 (y, t)_{t = t_0} = \psi (y) \tag{2.2}$$

where $a_1 = \text{const}$ and $a_2 (t)$ is a positive function which has a first order derivative.

Following [5] we shall seek a solution of (2.1) in the form function)

$$v_0 (y, t) = - \frac{2 a_2 (t)}{a_1} \frac{\partial}{\partial y} \ln \varphi (y, t) \tag{2.3}$$

where ($\varphi (y, t)$ is the new sough for function. To find this function we transform Eq. (2.1) in accordance with (2.4). First we obtain

$$\frac{\partial v_0}{\partial t} = \frac{\partial}{\partial y} \left[- \frac{2 a_2' (t)}{a_1} \ln \varphi - \frac{2 a_2 (t)}{a_1} \frac{\varphi_t'}{\varphi} \right] \tag{2.4}$$

$$\frac{\partial^2 v_0}{\partial y^2} = \frac{\partial}{\partial y} \left[- \frac{2 a_2'' (t)}{a_1} \frac{\varphi_{yy}''}{\varphi} + \frac{2 a_2 (t)}{a_1} \left(\frac{\varphi_y'}{\varphi} \right)^2 \right]$$

where the prime denotes a derivative with respect to the argument. From [5] it follows that (2.1) can be written in the form

$$\frac{\partial v_0}{\partial t} + \frac{\partial}{\partial y} \left[\frac{1}{2} a_1 v_0^2 - a_2 (t) \frac{\partial v_0}{\partial y} \right] = 0$$

In accordance with (2.4), we write this equation as follows:

$$\frac{\partial}{\partial y} \left[- \frac{2 a_2' (t)}{a_1} \ln \varphi - \frac{2 a_2 (t)}{a_1} \frac{\varphi_t'}{\varphi} \frac{2 a_2^2 (t)}{a_1} \frac{\varphi_{yy}''}{\varphi} \right] = 0$$

or

$$\frac{\partial \varphi}{\partial t} = a_2 (t) \frac{\partial^2 \varphi}{\partial y^2} - \frac{a_2' (t)}{a_2 (t)} \varphi \ln \varphi \tag{2.5}$$

The initial condition (2.2) for (2.5) has, with (2.4) taken into account, the form

$$\varphi(y, t)|_{t=t_0} = f_1(y)$$

$$f_1(y) = \exp\left\{-\frac{a_1}{2a_2(t_0)}\int_0^y \psi(\eta)\,d\eta\right\}, \quad t_0 = t_B$$

and the function $f_1(y)$ satisfies the Fourier conditions [8].

The equation (2.5) represents a nonlinear differential equation with variable coefficients. When $a_2(t) = a_2 =$ const, it yields a diffusion equation which, with an initial condition given, has a closed-type solution [5]. In [1] it was suggested that Eq. (2.1) be solved with $a_2 =$ const, and the value of a_2 found for each instant of time from the relations (1.4) or (1.5). Below we shall present the solution of (2.1) in a different form, with the variability of the coefficient $a_2(t)$ taken into account.

The solution of (2.5) has the form

$$\varphi(y, t) = \sum_k \exp\left\{\left[-\left(\frac{k\pi}{l}\right)^2 t + \frac{ik\pi y}{l_2} + \ln\varphi_0\right]\frac{1}{a_2(t)}\right\}, \quad \varphi_0 = \text{const} \tag{2.6}$$

Indeed, let

$$\Omega(y, t) = -\left(\frac{k\pi}{l}\right)^2 t + \frac{ik\pi y}{l_1} + \ln\varphi_0$$

from (2.6) we obtain

$$\frac{\partial\varphi}{\partial t} = \sum_k\left[-\left(\frac{k\pi}{l}\right)^2\frac{1}{a_2(t)} - \frac{a_2'(t)}{a_2^2(t)}\Omega\right]\exp\frac{\Omega}{a_2(t)} \tag{2.7}$$

$$\frac{\partial^2\varphi}{\partial y^2} = \sum_k\left(\frac{k\pi}{l}\right)^2\frac{1}{a_2^2(t)}\exp\frac{\Omega}{a_2(t)}$$

Substituting these relations and (2.6) into (2.1), we obtain an identity. In the case of cylindrical or spherical coordinates the solution of the Burgers' equation (1.3) has the form (2.3), (2.6), where $a_2(t)$ is determined from the formulas of the type (1.4) and (1.5), respectively. The required solution of (2.1) must satisfy the initial condition (2.2). From (2.2) and (2.6) we obtain for $t = t_0$

$$\varphi(y, t_0) = \sum_k \varphi_0^{1/a_2(t_0)} A_k \exp\frac{ik\pi y}{l_1} \tag{2.8}$$

$$A_k = \exp\left[-\left(\frac{k\pi}{l}\right)^2\frac{t_0}{a_2(t_0)}\right], \quad l_1 = la_2(t_0)$$

Let us now expand the function $f_1(y)$ into a complex Fourier series

$$f_1(y) = \sum_k \Phi_k \exp\frac{ik\pi y}{l_1}, \quad \Phi_k = \frac{1}{2l_1}\int_0^{2l_1} f_1(\eta)\exp\left(-\frac{ik\pi\eta}{l_1}\right)d\eta \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$\varphi_0 = (\Phi_k / A_k)^{a_2(t_0)} \tag{2.10}$$

Substituting (2.10) into (2.6), we obtain the solution of (2.1) with the initial condition (2.2). Now we can write the solution of (2.1) for the case of cylindrical and spherical coordinates.

Let us now turn our attention to certain facts emerging from the proposed solution of the Burgers' equation with variable coefficient. We recall the asymptotic solution for the diffusion equation with a constant coefficient (see [5]). In this solution the effect of the initial condition appears in the form of the principal moment of the initial perturbation. In the present case the solution consists of separate terms of the Fourier series and each term contains a constant which reflects the influence of the initial impulse. The number k of terms depends on the character of the initial condition. As $k \to \infty$, the solution obtained becomes exact. If the initial condition coincides exactly with one of the terms of the Fourier series, then the solution contains only this particular term. In the general case, the first term of the series gives only a rough approximation. The quantity l appearing in the expansion is taken as equal to the length of the impulse. When $a_2 = \text{const}$, the solution proposed also gives an accurate result and this can easily be confirmed by substituting the relations (2.7) into the corresponding diffusion equation and taking into account the fact that $a_2'(t) = 0$. However in this case the accuracy of the solution also depends on the number of terms in the expansion.

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